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A Characterization of Unimodular Orientations of Simple Graphs

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Let G be a simple graph, and consider an orientation of the edges of G. Where V is the vertex-set of G, let $A = (A_{vw} : v, w \in V)$ be the integral antisymmetric $(0, \pm 1)$-matrix such that $A_{vw} = 1$ if and only if vw is an edge of G directed from v to w. The orientation of G is said to be *unimodular* if for every $W \subseteq V$ the determinant of the submatrix $(A_{vw} : v, w \in W)$ is 0 or 1. For example, it is known that circle graphs can be provided with unimodular orientations.

For any vertex v and any subset of vertices W let $\varepsilon(v, W)$ be the number of edges leaving v and entering W less the number of edges entering v and leaving W. The main result of this paper says that the orientation of G is unimodular if and only if for every subset of vertices W such that $\varepsilon(w, W)$ is even for every $w \in W$, we can reverse the orientations of the edges which belong to some cocycle C so that $|\varepsilon(v, W)| \leqslant 1$ becomes true for every vertex. This generalizes a result of Ghouila-Houri for totally unimodular matrices.

1. INTRODUCTION

Let $A = (A_{vw} : v, w \in V)$ be an integral antisymmetric matrix indexed on a finite set V (thus $A_{vw} \in \mathbb{Z}$ and $A_{vw} = -A_{wv}$ for every $v, w \in V$). For each $W \subseteq V$ we define the *principal minor* $A[W] = (A_{vw} : v, w \in W)$. We are interested in the following property of unimodularity

$$\det(A[W]) \in \{0, 1\}, \qquad W \subseteq V, \tag{1.1}$$

which amounts to saying that every principal minor of A which is

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$$B' = \begin{array}{|c|c|} \hline 0 & B \\ \hline -B^t & 0 \\ \hline \end{array}$$

FIGURE 1

nonsingular has an integral inverse (recall that the determinant of an antisymmetric matrix is a square).

For example consider an arbitrary integral matrix B and construct the matrix B' depicted in Fig. 1. It is easy to verify that B' satisfies (1.1) if and only if B is totally unimodular. Thus (1.1) generalizes the concept of total unimodularity to those antisymmetric matrices which cannot be decomposed as in Fig. 1.

Our graphs will be simple. A graph is said to be *oriented* if an initial end and a final end are distinguished for every edge. The *adjacency matrix* associated to an oriented graph G is the integral antisymmetric $(0, \pm 1)$-matrix $A = (A_{vw} : v, w \in V(G))$ such that $A_{vw} = +1$ if and only if vw is an edge of G directed from v to w (and so $A_{vw} = -1$ if and only if wv is an edge of G directed from w to v). The orientation of G is said to be *unimodular* if A satisfies (1.1). Unimodular orientations are introduced in [2].

Not every graph G can be provided with an unimodular orientation. For example, one can verify that the 5-wheel (constructed from the 5-cycle C by adding a new vertex joined to every vertex of C) cannot be provided with an unimodular orientation. Suppose now that G is an oriented bipartite graph with color-classes X and Y, and define the $(0, \pm 1)$-matrix $B = (B_{xy} : x \in X, y \in Y)$ such that $B_{xy} = 0$ if no edge joins x to y, $B_{xy} = +1$ if there exists an edge oriented from x to y, $B_{xy} = -1$ otherwise. Then the adjacency matrix of G is decomposed like in Fig. 1. This implies that a (nonoriented) bipartite graph can be provided with an unimodular orientation if and only if it is the fundamental graph of a regular matroid (see [2] for details).

Let $W \subseteq V(G)$. We define the following notions for the oriented graph G: to *switch* the orientation at W is to reverse the directions of the edges which have precisely one end in W (which constitute the cocycle $\delta(W)$); the *degree excess* of a vertex v w.r.t. W, denoted $\varepsilon(v, W)$, is equal to the number of edges leaving v and entering W less the number of edges entering v and leaving W; W is a *weak independent* of G if $\varepsilon(w, W)$ is even for every $w \in W$ (weak independence obviously generalizes independence); the orientation is *balanced* w.r.t. W if $|\varepsilon(v, W)| \leqslant 1$ holds for every vertex v of G; the orientation is *perfectly balanced* if for every weak independent W we can switch the orientation at some subset $W' \subseteq W$ so that it becomes balanced w.r.t. W. Our main result is the following one.

(1.2) THEOREM. *An orientation of a simple graph is unimodular if and only if it is perfectly balanced.*

(1.3) COROLLARY (Ghouila-Houri [3]). *An integral matrix $B = (B_{xy} : x \in X,\ y \in Y)$ is totally unimodular if and only if every subset $Z \subseteq Y$ is the union of two disjoint subsets Z^+ and Z^- such that*

$$\text{(i)} \qquad \left| \sum (B_{xz} : z \in Z^+) - \sum (B_{xz} : z \in Z^-) \right| \leqslant 1, \qquad x \in X.$$

Proof (That 1.2 Implies 1.3). Let A be the antisymmetric matrix constructed from B like in Fig. 1 and let G be the oriented bipartite graph whose adjacency matrix is equal to A. The matrix B is totally unimodular if and only if the orientation of G is unimodular. Equality (i) is equivalent to saying that the following property holds after switching at Z^-:

$$\text{(ii)} \qquad |\varepsilon(x, Z)| \leqslant 1, \qquad x \in X.$$

If the orientation of G is actually unimodular then (1.2) implies that (ii) holds because Z is independent. Conversely suppose that (ii) holds and consider a weak independent $W \subseteq V = X \cup Y$. Consider the decomposition $W = T \cup Z$ with $T \subseteq X$ and $Z \subseteq Y$. Since Z is independent (and according weakly independent) there exists $Z^- \subseteq Z$ such that (ii) holds after switching at Z^-. Exchanging the roles of the rows and the columns, there exists $T^- \subseteq T$ such that after switching the new orientation at T^- we have

$$\text{(iii)} \qquad |\varepsilon(y, T)| \leqslant 1, \qquad y \in Y.$$

Note that property (ii) still holds after the second switching. Moreover the conjunction of (ii) and (iii) is equivalent to

$$\text{(iv)} \qquad |\varepsilon(v, W)| \leqslant 1, \qquad v \in V.$$

Thus (iv) holds after switching at $W' = T^- \cup Z^-$. (Note that we did not actually use the weak independency of W.) ∎

2. SUFFICIENT CONDITION OF THEOREM (1.2)

This condition is easy to prove. Moreover it simply implies the existence of an unimodular orientation for a circle graph. We already proved this result in [2] by constructing explicitely an integral inverse of every nonsingular principal minor. The new proof is nonconstructive.

Proof of the Sufficient Condition of (1.2). Let $A=(A_{vw}: v, w\in V(G))$ be the adjacency matrix of a graph G provided with a perfectly balanced orientation. Let $U\subseteq V(G)$ be such that $B=A[U]$ is nonsingular. We prove that B has an integral inverse. We denote by G' the subgraph induced by G on U.

The inverse of B is rational, thus we may consider an integral matrix $B'=(B'_{vw}: v, w\in U)$ such that BB' is an integral multiple of the identity matrix. Divide the terms of each column of B' by their greatest common divisor. This yields a new integral matrix $B''=(B''_{vw}: v, w\in U)$ such that

$$\text{(i)} \qquad BB''=D=(D_{vw}: v, w\in U),$$

with D an integral diagonal matrix and each column of B'' having at least one odd entry. Thus if we define $\mathrm{Odd}(w)=\{v\in U: B''_{vw} \text{ is odd}\}$, $w\in U$, then $\mathrm{Odd}(w)\neq\varnothing$.

Where $\bar{a}$ denotes the residue class mod 2 of an integer a, Equality (i) implies

$$\sum(\bar{B}_{uv}: v\in\mathrm{Odd}(w))=\sum(\bar{B}_{uv}\bar{B}''_{vw}: v\in U)=\bar{D}_{uw}, \qquad u, w\in U.$$

The inverse of the antisymmetric matrix B is also antisymmetric, so that $B'_{ww}=B''_{ww}=0$, and $w\notin\mathrm{Odd}(w)$. Thus for $u\in\mathrm{Odd}(w)$ we have $\bar{D}_{uw}=0$, which implies

$$\sum(\bar{B}_{uv}: v\in\mathrm{Odd}(w))=0, \qquad u\in\mathrm{Odd}(w), \quad w\in U.$$

And so $\mathrm{Odd}(w)$ is a weak independent of G' for every $w\in U$. Thus we can switch the orientation of G' on a cocycle $\delta(U_w)$, $U_w\subseteq U$, so that the new orientation becomes balanced w.r.t. $\mathrm{Odd}(w)$. We claim that the matrix $\beta=(\beta_{vw}: v, w\in U)$ defined by

$$\begin{aligned}
\beta_{vw}&=0 && \text{if } v\notin\mathrm{Odd}(w),\\
\beta_{vw}&=+1 && \text{if } v\in\mathrm{Odd}(w)\backslash U_w,\\
\beta_{vw}&=-1 && \text{if } v\in\mathrm{Odd}(w)\cap U_w,
\end{aligned}$$

is the inverse of B. For u, $w\in U$ we have

$$\sum(B_{uv}\beta_{vw}: v\in U)=\sum(B_{uv}B''_{vw}: v\in U) \qquad (\mathrm{mod}\ 2).$$

For $u\neq w$, the second summation is null by (i). This is also true for the first one because the orientation switched at U_w is balanced w.r.t. $\mathrm{Odd}(w)$. For $u=w$, the first summation takes its value in $\{-1, 0, +1\}$. It cannot be 0 since otherwise the vector $\beta_w = (\beta_{vw} : v \in U)$ would be in the kernel of $A[U]$, which implies $\beta_w = 0$ because B is nonsingular, a contradiction with $\mathrm{Odd}(w) \neq \varnothing$. Thus after possibly changing the signs in β_w, we have

$$\sum (B_{uv}\, \beta_{vw} : v \in U) = \delta_{uw},$$

where δ_{uw} denotes the Kronecker symbol, which implies that β is the inverse of B. ∎

Let m be a word such that each letter occuring in m occurs precisely twice. We say that m is a *double occurrence word.* An *alternance* of m is a nonordered pair $v'v''$ of distinct letters such that we meet alternatively $\cdots v' \cdots v'' \cdots v' \cdots v'' \cdots$ when reading m. The *alternance graph* $G(m)$ is the simple graph whose vertices are the letters of m and whose edges are the alternances of m. From a geometric point of view, alternance graphs can be interpreted as intersection graphs of chords of a circle, and they are more widely known as *circle graphs.*

A *separation* of m is any word $(v_1, \varepsilon_1)(v_2, \varepsilon_2) \cdots (v_{2n}, \varepsilon_{2n})$, over the set of letters $V \times \{-1, +1\}$, such that $m = v_1 v_2 \cdots v_{2n}$ and $(v_p, \varepsilon_p) \neq (v_q, \varepsilon_q)$ for $1 \leqslant p < q \leqslant 2n$. Let $v'v''$ be an alternance of m, and let $\cdots (v', s') \cdots (v'', s'') \cdots (v', -s') \cdots (v'', -s'') \cdots$ be the succession of the letters belonging to $\{v', v''\} \times \{-1, +1\}$, then the edge $v'v''$ of $G(m)$ will be directed from v' to v'' if $s' = s''$, from v'' to v' otherwise. This orientation of $G(m)$, first introduced by W. Naji [4], will be called a *Naji orientation.*

(2.1) PROPOSITION. *Naji orientations are perfectly balanced.*

Proof. Consider a double occurrence word $m = v_1 v_2 \cdots v_{2n}$, the alternance graph $G = G(m)$, a Naji orientation induced by a separation $\mu = (v_1, \varepsilon_1)(v_2, \varepsilon_2) \cdots (v_{2n}, \varepsilon_{2n})$. Let W be a weak independent of $G(m)$, and let $m' = v_{i_1} v_{i_2} \cdots v_{i_{2q}}$ be the subword of m obtained by deleting every letter not belonging to W. Define $\varepsilon'_i \in \{-1, +1\}$, $1 \leqslant i \leqslant 2n$, in the following way: (i) if $v_i \notin W$ then $\varepsilon'_i = \varepsilon_i$, (ii) if $v_i \in W$ and $i = i_k$, $1 \leqslant k \leqslant 2q$, then $\varepsilon'_i = (-1)^k$. We claim that $\mu' = (v_1, \varepsilon'_1)(v_2, \varepsilon'_2) \cdots (v_{2n}, \varepsilon'_{2n})$ is another separation of m. Indeed let $v \in V$, and let i and j be the indices such that $1 \leqslant i < j \leqslant 2n$ and $v = v_i = v_j$. In case (i) we have $\varepsilon'_i = \varepsilon_i = -\varepsilon_j = -\varepsilon'_j$. In case (ii) v_i and v_j occur in m' and we can assert that an even number of letters occur in m' between v_i and v_j because the degree of v in $G[W]$ is even. So μ' is a separation of m. Next we show that the corresponding Naji orientation is perfectly balanced. The successive elements $\varepsilon'_{i_1} \varepsilon'_{i_2} \cdots \varepsilon'_{i_{2q}}$ alternate in sign. Therefore the number of these elements with a positive sign having an index i_s such that $i < i_s < j$ differs by at most 1 from the number of these elements with a negative sign belonging to the same interval. This implies that $G(m)$ provided with the Naji orientation induced by μ' is balanced w.r.t. W. Finally, where W' is the subset of the elements of W paired with distinct signs in μ and μ', we note that the Naji orientation induced by μ' is obtained by switching at W' the Naji orientation induced by μ. ▮

(2.2) COROLLARY. *Naji orientations are unimodular.*

Proof. This is directly implied by (2.1) and the sufficient condition of (1.2). ▮

3. NECESSARY CONDITION OF THEOREM (1.2)

To prove the necessary condition we use Tutte's representation of matroids by chain groups, a method which could also be used to derive shortly Ghouila-Houri's theorem from Tutte's results on primitive chains of a regular matroid [5]. We already used chain groups to study the representations of Δ-matroids and symmetric matroids over a field [1]. To make this paper self-contained we do not use explicitely Δ-matroids and we recall the proof of Lemma (3.1) already given in [1]. On the other side we assume that the reader is familiar with the basic facts of matroid theory.

Let R be an integral domain, and let V be a finite set. We consider $R^V = \{A = (A_v : v \in V) : A_v \in R\}$. Any element of R^V is called a *chain* (on R over V). The *support* of a chain A is $\|A\| = \{v \in V : A_v \neq 0\}$. If $R = \mathbb{Z}$ the *odd support* of A is $\{v \in V : A_v$ is odd$\}$. Any subspace N of R^V is called a *chain-group* (on R over V). An *elementary* chain of N is a nonnull chain of N with a minimal support. The supports of the elementary chains are the circuits of a matroid denoted by $M(N)$.

Let $A = (A_{sv} : s \in S, v \in V)$ be a matrix with entries in R. For any $T \subseteq S$ and $W \subseteq V$ we denote by $A[T, W]$ the submatrix of A whose rows and columns are respectively indexed by T and W (and we still use the notation $A[W]$ for a principal minor in the case $S = V$). A is called a *representative matrix* of N if its rows belong to N and its rank is equal to the dimension of N. Then a subset $W \subseteq V$ is a cobase of $M(N)$ if and only if the submatrix $A[S, W]$ is a square submatrix with a nonnull determinant.

We shall consider the following situation. Starting with an antisymmetric matrix $A = (A_{vw} : v, w \in V)$ with coefficients in R, we consider a bijection $v \to v^\sim$ from V into a disjoint set $V^\sim$, we let $V' = V \cup V^\sim$, and for every $w = v^\sim \in V^\sim$ we let $w^\sim = v$, so that the mapping $v \to v^\sim$ is an involution over V'. We construct the matrix $A' = (A'_{vw} : v \in V,\ w \in V')$ defined by $A'[V, V] = A$, $A'_{vw^\sim} = 0$ for $v \in V$ and $v \neq w \in V$, $A'_{vv^\sim} = 1$ for $v \in V$. We call A' a *standard extension* of A. A subset $T \subseteq V'$ is called a *subtransversal* (*transversal*) if $|T \cap \{v, v^\sim\}| \leqslant 1$ ($|T \cap \{v, v^\sim\}| = 1$) for every $v \in V$. Each pair $\{v, v^{\sim}\}$ is called a *symmetric pair* of V'. For $P \subseteq V$ we let $P^{\sim} = \{v^{\sim} : v \in P\}$.

(3.1) LEMMA. *We use the above notation and we let N be the chain-group generated by the rows of A'. The two following properties hold:*

(3.1.1) *every subtransversal independent set of $M(N)$ is included in a transversal base of $M(N)$;*

(3.1.2) *if $R = \mathbb{Z}$ then no circuit of $M(N)$ includes precisely one symmetric pair.*

Proof. We define over $R^{V'}$ the bilinear form

$$(\gamma, \delta) \to \gamma\delta = \sum (\gamma_v \delta_{v^{\sim}} : v \in V').$$

Denote by A'_v the v-row of A', $v \in V$. For $v, w \in V$, we have

$$A'_v A'_w = A'_{vv^{\sim}} A'_{wv} + A'_{vw} A'_{ww^{\sim}} = A'_{wv} + A'_{vw}$$

because $A'[V, V^{\sim}]$ is an identity matrix. The right member is null because A is antisymmetric. Thus any two, possibly equal, chains of N are orthogonal.

If J is a subtransversal independent set of $M(N)$, and $\{v, v^{\sim}\}$ is a symmetric pair disjoint from J, we claim that either $J \cup \{v\}$ or $J \cup \{v^{\sim}\}$ is a subtransversal independent set, too. Indeed if it is false we can find two chains γ' and γ'' in N with supports included in $J \cup \{v\}$ and $J \cup \{v^{\sim}\}$, respectively. We have

$$\gamma'\gamma'' = \gamma'_v \gamma''_{v^{\sim}} \neq 0$$

because if either $\gamma'_v = 0$ or $\gamma''_{v^{\sim}} = 0$, then J is not independent. Thus we can, step by step, augment the independent J to a transversal base of $M(N)$, which proves (3.1.1).

If the support of a chain γ includes precisely one symmetric pair $\{v, v^{\sim}\}$, we have

$$\gamma\gamma = 2\gamma_v \gamma_{v^{\sim}} \neq 0 \text{ because } R = \mathbb{Z},$$

so that $\gamma \notin N$, which proves (3.1.2). ∎

If $R = \mathbb{Z}$ an elementary chain with coefficients in $\{-1, 0, +1\}$ is called a *primitive chain.* A *reduction mod* 2 of a chain f is a chain f' with coefficients in $\{-1, 0, +1\}$ such that $f(v) = f'(v)$ (mod 2) for every $v \in V$. The following theorem implies (1.2).

(3.2) THEOREM. *Let $A=(A_{vw}: v, w \in V)$ be an antisymmetric matrix over $\mathbb{Z}$ and $A'=(A'_{vw}: v \in V, w \in V')$ be a standard extension of A. Where N is the chain group generated by the rows of A', the following properties are equivalent*:

(i) *every principal minor of A has a determinant equal to* 0 *or* 1;

(ii) *every square submatrix of A' whose set of columns is indexed by a transversal has a determinant equal to* 0 *or* ± 1;

(iii) *every subtransversal circuit of $M(N)$ is the support of a primitive chain*;

(iv) *for every chain $f \in N$ whose odd support is subtransversal there exists $f' \in N$ which is a reduction of f* mod 2;

(v) *the orientation of the graph G defined by the adjacency matrix A is perfectly balanced.*

Proof. We already proved (v) $\Rightarrow$ (i) which is the sufficient condition of Theorem (1.2). Thus we prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

(i) $\Rightarrow$ (ii). For every transversal W'' of V' there exists a subset $W \subseteq V$ such that $W''=W \cup (V \backslash W)^{\sim}$. Property (ii) is implied by the following equalities:

$$\det A'[V, W'']= \pm \det A'[W, W]= \pm \det A[W].$$

(ii) $\Rightarrow$ (iii). Note that if γ and γ' are elementary chains of N, with $\|\gamma\|=\|\gamma'\|$, then there exists $\alpha, \alpha' \in \mathbb{Z}$ such that $\alpha\gamma=\alpha'\gamma'$. Thus to prove (iii) it is sufficient to find for each subtransversal circuit C of $M(N)$, and each $v \in C$, a chain $\gamma \in N$ whose v-coefficient is equal to 1 and such that $\|\gamma\|=C$. The subset C-v is a subtransversal independent, and so Lemma (3.1) implies the existence of a transversal base Z such that C-$v \subseteq Z$. We consider the transversal cobase $Y=V' \backslash Z$. The square submatrix $A'[V, Y]$ is non-singular, its determinant is equal to ± 1 by (ii), and so it has an integral inverse, say $P=(P_{yv}: y \in Y, v \in V)$ such that the product $B'=PA'$ satisfies $B'[Y, Y]=(\delta_{yz}: y, z \in Y)$, where δ_{yz} is a Kronecker symbol. Thus B' is a standard matrix. Its rows are elementary chains of N. Note that $v \notin Z$ and consider the row $B'_v=(B'_{vv'}: v' \in V')$. We have $B'_{vv}=1$ and $\|B'_v\|$ is the unique circuit of $M(N)$ included in $Z \cup \{v\}$. Thus $\|B'_v\|=C$ and we can take $\gamma=B'_v$.

(iii) $\Rightarrow$ (iv). For $\gamma \in N$ let $\bar{\gamma}$ be the chain on $GF(2)$ over V' such that $\bar{\gamma}(v)$ is the residue class (mod 2) of $\gamma(v)$ for every $v \in V'$, and let $\bar{N}=\{\bar{\gamma}: \gamma \in N\}$.

CLAIM. Every subtransversal circuit of $M(\bar{N})$ is the support of a primitive chain of N.

Proof of the claim. Let C be a circuit of $M(\bar{N})$ and let $\bar{\gamma}$ be an elementary chain of N such that $\|\bar{\gamma}\| = C$. Consider any $v \in C$, the independent C-v, a transversal base Z of $M(\bar{N})$ including C-v (which exists by (3.1.1) applied with $R = GF(2)$) and the transversal cobase $Y = V' \backslash Z$ of $M(\bar{N})$. The binary matrix $\bar{A}'$ obtained by replacing each entry of A' by its residue class mod 2 is a representative matrix of $M(\bar{N})$. Therefore $\bar{A}'[V, Y]$ has a nonnull determinant in $GF(2)$. This implies that $A'[V, Y]$ has an odd determinant, so that Y is a cobase of $M(N)$ and Z is a base of $M(N)$. There exists a circuit D of $M(N)$ included in $Z \cup \{v\}$, and this circuit is subtransversal since otherwise $\{v, v^{\sim}\}$ would be the unique symmetric pair included in D, a contradiction of (3.2.2). Following (iii), D is the support of a primitive chain δ of N. The support of $\bar{\delta}$ is also equal to D. Therefore there exists a circuit D' of $M(\bar{N})$ such that $v \in D' \subseteq D$. The circuit D', like C, is included in $Z \cup \{v\}$, and so it is equal to C. This implies $D = C$. ∎

Let γ be chain of N with a subtransversal odd support C. The support of $\bar{\gamma}$ is equal to C. Since $\bar{N}$ is a binary chain-group we can decompose $\bar{\gamma}$ into a sum of elementary chains of $\bar{N}$, $\bar{\gamma}_1 + \bar{\gamma}_2 + \cdots + \bar{\gamma}_k$ say, having pairwise disjoint supports. Following the claim we can find a primitive chain $\gamma_i \in N$ having the same support as $\bar{\gamma}_i$, $1 \leqslant i \leqslant k$. The chain $\gamma_1 + \gamma_2 + \cdots + \gamma_k$ is a reduction mod 2 of γ.

(iv) $\Rightarrow$ (v). For every $w \in V$ we denote by A'_w the row of A' indexed by w. Let W be a weak independent of G, and let $\gamma = \sum (A'_w : w \in W)$. γ is a chain of N whose support is included in $W^{\sim} \cup V$. Since W is weakly independent γ_w is even for every $w \in W$, which implies that the odd support of γ is subtransversal. Following (iv) we can find a chain γ' which is a reduction of γ mod 2. The equality $\gamma_{w^{\sim}} = +1$ implies $\gamma'_{w^{\sim}} = \pm 1$ for every $w \in W$. Let $W^+ = \{w \in W : \gamma'_{w^{\sim}} = +1\}$ and $W^- = \{w \in W : \gamma'_{w^{\sim}} = -1\}$. Since A' is a standard matrix we have

$$\gamma' = \sum (A'_w : w \in W^+) - \sum (A'_w : w \in W^-).$$

Let B' be the matrix obtained by changing the sign of each row A'_w, $w \in W^-$. Then we have

$$\gamma' = \sum (B'_w : w \in W).$$

Finally let B'' be the matrix obtained by changing the signs of the columns of B' indexed on W^-. The submatrix $B''[V, V]$ is the adjacency matrix of the graph G after switching its orientation at W^-. In this new directed graph the degree-excess from any vertex v to W is equal, up to the sign, to γ'_v, which proves the property since γ' is reduced mod 2. ∎

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